THURSDAY, NOVEMBER 30, 1905.

LAGUERRE'S MATHEMATICAL PAPERS.

Œuvres de Laguerre. Tome ii. Pp. 715. (Paris: Gauthier-Villars, 1905.) Price 22 francs.

THE publication of the mathematical papers of Laguerre, undertaken after his death in 1886 by MM. Hermite, Poincaré, and Rouché under the auspices of the Academy of Sciences, has at length been completed. A first volume, to which M. Poincaré contributed as a preface an admirable appreciation of the author, appeared in 1898; and now some eighty papers which treat of geometrical subjects have been collected from the various scientific journals and reprinted in a second and final volume of more than 700 pages. Most of these papers are of but four or five pages in length, for it was Laguerre's habit, when a mathematical investigation had aroused his interest, to return to it again and again as new ideas occurred to him; and so it comes to pass that the majority of his writings on geometry may be classified as dealing with one or other of some half-dozen wide but distinct subjects.

The discovery with which Laguerre made his entry into the ranks of original investigators is of such moment in the history of modern mathematics that we will pause in order to realise the condition of geometrical knowledge at the time, and the circumstances in which it was made. The first three of the papers now under review bear the dates 1852–3, and were written when the author, a student eighteen years of age at the Institution Barbet, was still only a candidate for admission to the École Polytechnique. It was a time when a great change in geometrical thought had been initiated. Poncelet and Chasles had begun to build up the theory of projective geometry, and through it mathematicians had been made to recognise that theorems previously regarded as wholly without connection might in reality be but different presentments of the same more fundamental fact. But the structure was as yet very far from complete; many of the chief features of the theory as we now know it were still obscure and needed explanation. It fell to Laguerre to provide the most important and prolific discovery on which modern geometrical theory has been founded, the inner meaning of angular magnitude. In the second of his three early papers was enunciated for the first time the proposition that the sides of an angle form with the two isotropic lines through the vertex a pencil the anharmonic ratio of which depends only upon the magnitude of the angle.

This was no chance discovery, lighted on by a stroke of undeserved good fortune, for the rest of the paper shows how true a grasp of the new principles Laguerre had already obtained. He goes on to point out that the proposition furnishes the solution of the problem of homographic transformation of angular relations—a problem which had baffled the founders of projective geometry—and gives many further developments and results, a special case of one of which may be cited as illustrative. The well known theorem of Menelaus concerning [illegible]s of the segments into which the sides of a triangle are divided by any straight line is identical with the theorem that the angles of a triangle make up two right angles; either theorem can be deduced from the other. It is regrettable that English treatises upon analytical geometry so rarely attribute theorems to their authors, for it is with this discovery, rather than any achieved later in more advanced subjects, that we should wish the name of Laguerre to be always associated. Certainly no discovery has had so far-reaching an influence upon geometrical research during the past half-century.

In the twelve years which followed Laguerre published nothing. His military duties as an officer of artillery at first absorbed him; then, having been transferred to the manufactory of arms at Mutzig, he found leisure to take up once more his favourite study; it was not, however, until 1865, after his recall to Paris to the École Polytechnique, that he published the first of the series of original papers which he continued without intermission until his death.

A large number of these are concerned with analytical geometry, and through an interesting section of them runs an idea allied to his earlier work, the use of imaginaries in geometry. Thus in one of the first papers we find a full exposition of the theory of foci as extended from conics to plane curves of any class, and the distinction is drawn between ordinary foci and singular foci of a curve which passes through the circular points. Another such matter with which Laguerre frequently occupied himself, and which seems never to have received the attention it deserves, is the means of representing in a concrete manner the points of a plane or in space the coordinates of which are complex quantities. Imaginary values of the coordinates may satisfy the equation of a curve, and are then often spoken of as the coordinates of an imaginary point on the curve; cannot geometry suggest some mode of representing these points similar to that of Argand, which plays so important a part in the theory of equations involving a single variable? No solution could be satisfactory which varied with the coordinate-system employed. Laguerre's method was to represent a pair of points the coordinates of which are conjugate complex quantities by a real segment of a line, distinguishing the two imaginary points when necessary by the sense in which the segment is measured; the ends of the segment are the intersections of the lines which join the imaginary points to the circular points. Thus the line joining two real foci of a conic represents the two imaginary foci. It now becomes feasible to express the conditions of collinearity, &c., of imaginary points by properties of their representative segments. For points in space a similar procedure is adopted; a point the coordinates of which are complex is represented by a real circle, which must be described in a definite sense.

Having extended the notion of a focus to curves other than conics, it was natural that Laguerre should study curves which possess focal properties; accord

ingly he made many investigations upon special quartic curves, both plane and twisted; he considered the meaning of a focus of a curve traced on a sphere, showing that the method of inversion changed the ordinary foci of such a curve into foci of the inverse curve; he published many investigations upon a class of loci to which at the time considerable attention was paid, anallagmatic curves and surfaces. Other papers treat of the cyclide of Dupin, the curve of intersection of two quadric surfaces, and a family of curves called by Laguerre Cassinians: and it is to be noted that while writing upon a particular curve he would at times include theorems of wider application.

A passing mention may be accorded to an interesting statement of the addition theorem of hyperelliptic functions closely resembling that derived from Poncelet's polygons in elliptic functions; to researches upon Steiner's Roman surface and its reciprocal, known as Cayley's cubic surface, and other applications of the theory of forms to geometry. Finally, at the end of the book we meet with a series of papers in which Laguerre's discovery of *geometry of direction* is developed. The idea from which this sprang is elementary enough; a straight line or a circle may be traced out by a moving point in two opposite senses, and therefore is regarded by Laguerre as composed of two "half-lines" or two cycles. The notion of tangency is modified when a curve is described in a definite direction, so that a cycle is regarded as possessing one tangent only parallel to a given half-line. Following up this thought, Laguerre is led to divide all curves into curves of direction, which can be divided analytically into two trajectories traced out in different senses, and curves which have not this property; he finds the form of the tangential equation of the most general curve of direction. By help of a highly ingenious "transformation by reciprocal half-lines," it is shown how certain problems may be greatly simplified; the problem of drawing a circle to touch three given circles, for example, is reduced to that of drawing a circle through three points. The theory is extended also to spherical geometry.

Laguerre's life-work in geometry forms a volume which no mathematician can study without being profoundly impressed by the ingenuity of the author and his skill in handling every method which he employs; papers such as his, models of clear polished style, are read with keen intellectual enjoyment. Yet when the book is laid down and we reflect on the work as a whole, there comes a regretful conviction that what has been accomplished is very far from all that could have been hoped for from the powers of the author and his brilliant first achievement. Delicate in health and of retired life, Laguerre's isolation from the march of scientific thought is betrayed in his writings. General notions appealed to him solely by their applicability to particular problems, and he therefore chose to bestow the utmost care upon a number of short discussions of special topics. Let the reader, if he would appreciate what is best in these collected writings of Laguerre, realise when he takes up the book that its author was one of those who are content to apply to small things' powers capable of far higher work, and he will find matter to arouse his interest and admiration in every paper reprinted in the volume.

PHILOSOPHICAL STUDIES.

(1) *Goethe's Philosophie aus seinen Werken.* Edited with an introduction, by Max Heynacher. Pp viii+428. (Leipzig: Dürr'sche Buchhandlung 1905.) Price 3.60 marks.

(2) *Immanuel Kant, Physische Geographie.* Second edition. Edited by Paul Gedan. Pp. xxx+386. (Leipzig: Dürr'sche Buchhandlung, 1905.) Price 2.50 marks.

(3) *Dialoge über natürliche Religion, über Selbstmord und Unsterblichkeit der Seele.* By David Hume Translated into German and edited by Dr. F Paulsen. Third edition. Pp. 165. (Leipzig Dürr'sche Buchhandlung, 1905.) Price 1.50 marks

(4) *Immanuel Kant's Kleinere Schriften zur Logik und Metaphysik.* Second edition. Edited by Karl Vorländer. In four parts. Pp. xxxii+169, xl+172, xx+175, xxxi+176. (Leipzig: Dürr'sche Buchhandlung, 1905.) Price 5.20 marks.

(5) *G. W. F. Hegel, Encyclopädie der philosophischen Wissenschaften im Grundrisse.* Second edition Edited by Georg Lasson. Pp. lxxvi + 522 (Leipzig: Dürr'sche Buchhandlung, 1905.) Price 3.60 marks.

(1) GOETHE'S work was so many-sided, and withal so voluminous, that it is a real service to educated thought to have presented, as here, a volume of extracts, in moderate compass, containing in his own words an account of the great writer's philosophic and scientific views, and of the influences exerted on him by different systems. Herder, Spinoza and Kant all obviously attracted him at various times, and his name must find a place in any account of the theory of colour or of comparative anatomy—to name only two of the scientific subjects in which he was interested. With these and kindred matters the editor deals in a well-informed introduction. He knows the literature well, his Eckermann, the Goethe Jahrbuch, and Goethe's poetry. Goethe's title to be regarded as a forerunner of Darwin is duly emphasised.

(2) That Kant should thus have lined the wings of his spirit in the dregs of the sensible world will astonish the average reader, for this work condescends to minute details regarding the animal, vegetable, and mineral kingdoms, the characteristics of different races of men, and the like. Even one of the earlier parts, dealing with mathematical preliminaries, is not at all speculative in its nature, and only one or two paragraphs in the introduction, which point out that geography deals with facts in space as history with events in time, remind us of the Critique of Pure Reason; but the services of Kant to geography are not negligible, and have been attested by Helmholtz.

The present edition contains a full statement of